ALGEBRA II

Mid-Term Examination
24 February 2012

Instructions: All questions carry equal marks.

(1) Let R be a commutative ring with unity and let I be an ideal of R.
   (a) Prove that the quotient abelian group R/I has a ring structure so that the natural group homomorphism from R to R/I becomes a ring homomorphism.
   (b) Prove that the ideals of the ring R/I are in one-one correspondence with ideals of R that contain I.

(2) Let $\mathbb{Z}[X]$ denote the ring of polynomials with integer coefficients. Prove that:
   (a) The ideal generated by the polynomial $X^2 + 2X + 1$ is not a prime ideal.
   (b) The ideal generated by $X^2 + 3X + 1$ is a prime ideal which is not a maximal ideal.

(3) Let $C[0, 1]$ denote the real vector space of continuous real valued functions on the closed interval $[0, 1]$. Show that $\{x^3, sin(x), cos(x), e^x\}$ is a linearly independent subset of $C[0, 1]$.

(4) Let V be a vector space over a field F and let S be a spanning subset of V. Prove that S contains a basis of V. (Note that S is not assumed to be finite or countable!)

(5) Determine the dimensions of the kernel and the image of the linear operator T on the vector space F^n defined by:

$$T(x_{1,}x_2, \ldots, x_n) = (x_1 + x_n, \ldots, x_i + x_{n-i+1}, \ldots, x_n + x_1)$$